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Motions of Passenger Cars in Low-Speed Falls Over Embankments

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ABSTRACT: An analysis of the dynamics of motions of passenger cars in falls over embankments at speeds less than that required to launch a car into a free-fall trajectory is developed in this paper as an aid in evaluating initial vehicle speed in this particular type of crash involved in the forensic science field of accident reconstruction.

KEYWORDS: engineering, automobiles, motor vehicle accidents, collision research

Nomenclature

a Linear acceleration
B Wheelbase
b Distance from mass center to rear wheels
c Distance from mass center to front wheels
d Distance from front wheels
L Distance from mass center to landing contact point on vehicle
F Linear force
g Acceleration of gravity
h Distance ground to underside of vehicle
h_c Distance ground to mass center
I Moment of inertia
M Mass
S Linear displacement
t Time
T Torsional moment
V Velocity
W Weight
x x direction, Cartesian coordinate
y y direction, Cartesian coordinate
α Angular acceleration
ω Angular velocity
θ Angular displacement

Angular values are in radians except when used in trigonometric functions.

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Introduction

In the event a passenger car falls over an embankment at speeds less than that which would produce a free-fall launch motion, several modes of motion result depending on the initial speed, on friction factors, and on the physical properties of a given vehicle. Consideration is given in the following discussion to some possible types of motion of passenger cars falling over an embankment as an aid in evaluating car speeds in the forensic science field of accident reconstruction.

Certain assumptions have been made in this paper to simplify the presentation; that acceleration is constant for a given time period, that the effect of small slopes of the ramp surface is negligible, and that the terminus of the ramp surface is sharp-edged. It is assumed that the initial motion of the car is perpendicular to the edge of the ramp surface so both front wheels move over the edge of the ramp in a pitching motion at the same time. In actual cases, the car may be at an angle to the edge of the embankment and may also be in a lateral motion so the front and rear wheels on one side leave the embankment simultaneously in a roll motion of the car. While the initial assumptions are somewhat idealistic, it is often the practice in developing engineering analyses that the theoretical approach be tempered by judgment or substantiated by test.

The maximum velocity considered in this presentation is the minimum velocity an automobile would need to achieve a free-fall launch motion as treated in Ref *1*. It can be seen upon substitution of numerical values for a given vehicle in the minimum launch velocity equation the range of velocities involved in this analysis is comparatively small. However, in practical problems it is worthwhile to be able to account for pitch displacements and times of motion of a car in certain circumstances after it has exited the embankment.

While this presentation is somewhat elemental, it is hoped that it will engender comments and contributions to much needed improvements in the theory and practice of accident reconstruction.

Range of Vehicle Velocities Involved in Falls Over Embankments

Assume that a car is put into a position as diagramed in Fig. 1 and allowed to fall to a contact between the underside of the car and embankment when the front wheel support is suddenly removed.

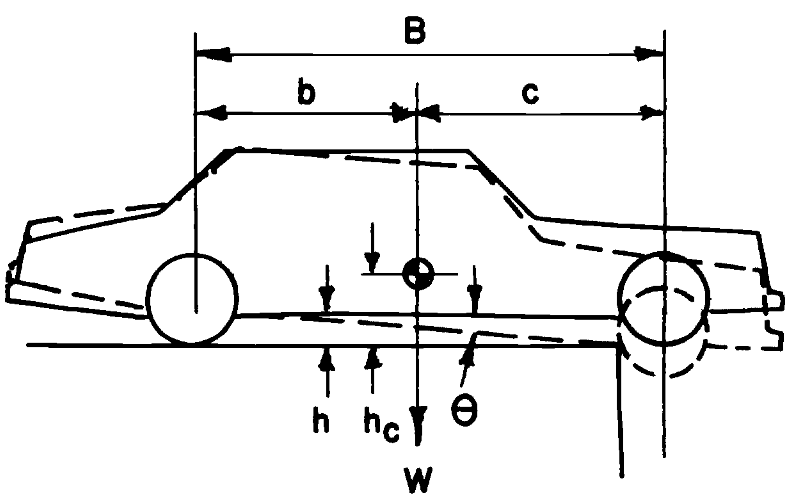

FIG. 1—*Simple drop of front wheels over an embankment,* V = *0.*

Summing forces in the y direction,

$$\Sigma F_y = Ma_y$$

$$\Sigma F_y = W - W_R = W - \frac{Wc}{B} = W\left(1 - \frac{c}{B}\right)$$

$$\frac{Wa_y}{g} = W\left(1 - \frac{c}{B}\right)$$

so the acceleration of the mass center in the vertical direction is

$$a_y = g(1 - c/B) \tag{1}$$

Summing moments about the rear wheels

$$\Sigma T = Wb = I\alpha + Ma_y b$$

$$= I\alpha + \frac{Wa_y b}{g}$$

so the angular acceleration is

$$\alpha = \frac{Wb - Wa_y b/g}{I} = \frac{Wb(1 - a_y/g)}{I} = \frac{Wbc}{BI} \tag{2}$$

The angular displacement is

$$\theta = \frac{h}{B - d} \tag{3}$$

Consider a front wheel in some position as diagramed in Fig. 2 as it rolls over the edge of an embankment. For a given velocity of the car V, the time to travel the horizontal distance S_h, is

$$t_h = \frac{S_h}{V} = \frac{r \sin \beta}{V} \tag{4}$$

in which

$$\beta = \arccos \frac{r - h}{r} \tag{5}$$

The corresponding vertical distance S_v is

$$S_v = r - r\cos\beta = r(1 - \cos\beta)$$

and the angular displacement of the car is

$$\theta = \frac{S_v}{B - r\cos\beta} = \frac{r(1 - \cos\beta)}{B - r\cos\beta}$$

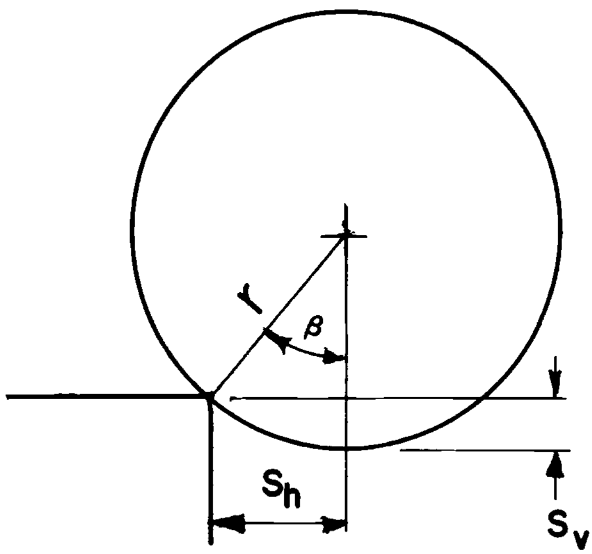

FIG. 2—*Wheel on edge of embankment.*

and by Eq 2 the time to move a distance S_v is

$$\frac{Wbct_v^2}{2BI} = \frac{r(1 - \cos \beta)}{B - r \cos \beta}$$

from which

$$t_v = \sqrt{\frac{2BIr(1 - \cos \beta)}{Wbc(B - r \cos \beta)}} \tag{6}$$

When the underside of the car contacts the edge of the embankment behind the front wheels, $t_h = t_v$ and

$$\frac{r^2 \sin^2 \beta}{V^2} = \frac{2BIh}{Wbc(B - r \cos \beta)}$$

from which the minimum velocity is

$$V = \sqrt{\frac{Wbcr^2(B - r \cos \beta) \sin^2 \beta}{2BIh}} \tag{7}$$

Between the angles $\beta = 0$ and $\beta = \text{arc cos } [(r - h)/r]$, the time t_h by Eq 4 is larger than t_v by Eq 5 which means that for a minimum velocity expressed by Eq 7 the wheels remain in contact with the edge of the embankment until the underside of the car contacts the embankment.

For an alternate solution, the minimum velocity of the car when the underside contacts the embankment can be approximated by assuming that the change in potential energy in the falling motion is converted into linear velocity, that is,

$$W\Delta h_c = \frac{WV^2}{2g}$$

from which

$$V = \sqrt{2g\Delta h_c} \tag{8}$$

where

$$\Delta h_c = \frac{bh}{B - r\sin\beta}$$

It has been shown in Ref *1* that if the initial velocity is

$$V = \sqrt{\frac{4WB^2bc}{54Ih}} \tag{9}$$

the underside of the car will contact the edge of the embankment without modifying the velocity or pitch motion and the vehicle will launch into a free-fall motion. The value of velocity by Eq 7 represents the minimum value, while the velocity obtained by Eq 9 represents the maximum value of velocity considered in this motion analysis.

Underside Contact Distance d *as a Function of Velocity*

As the velocity of a car increases above a minimum as expressed by Eq 7, the forward displacement exceeds the fall displacement so the front wheels separate from the edge of the embankment. The separation point on the wheel can be expressed by the angle β derived from Eqs 4 and 6, that is,

$$\frac{r^2\sin^2\beta}{V^2} = \frac{2BIr(1 - \cos\beta)}{Wbc(B - r\cos\beta)}$$

and

$$V = \sqrt{\frac{Wbcr\sin^2\beta\,(B - r\cos\beta)}{2BI(1 - \cos\beta)}} \tag{10}$$

Upon substituting numerical values in Eq 10, it can be seen that a separation of the wheel from the edge of the embankment can occur at small values of β with only a small increase of velocity above the minimum given by Eq 7. If the velocity is approximately 20% greater than the minimum velocity, it can be assumed that the motion after the front wheels pass over the edge of the embankment is simply one of free-fall until the underside of the car contacts the edge of the embankment.

By Eq 2, the acceleration of pitch motion is

$$\alpha = \frac{Wbc}{BI}$$

so the angular velocity of pitch motion is

$$\omega = \frac{Wbcd}{BIV} \tag{11}$$

and the pitch displacement at time $t = d/V$ is

$$\theta = \frac{\alpha t^2}{2} = \frac{Wbcd^2}{2BIV^2} \tag{12}$$

then,

$$h = \theta(B - d) = \frac{Wbc(B - d)d^2}{2BIV^2}$$

from which

$$V = \sqrt{\frac{Wbcd^2(B - d)}{2BIh}} \tag{13}$$

In practical cases, the distance d can be measured and substituted in Eq 13 with the physical properties of the vehicle to give the velocity at the time the front wheels passed over the edge of the embankment.

Postcontact Motions

Once the underside of the car has fallen to contact the embankment, different conditions of motion may exist:

1. The forward velocity of the car may be low enough and the drag forces between the underside of the car and the embankment may be large enough to stop the forward motion within a distance d less than the distance c.
2. The forward velocity and drag forces may be such that the car comes to rest at a distance d greater than the distance c.
3. The forward velocity and drag forces may be such that the distance d is greater than c, but the car does not come to rest.
4. The embankment may be high enough that in the postcontact falling motion the rear wheels strike the embankment.
5. The embankment may be high enough that the car can go into a free-fall stage before landing.
6. The car may pitch over to come to rest on its top.

These conditions are treated in the following.

Velocity Reaches Zero at d *Less Than* c

Considering the case in which a car contacts the embankment at a distance d_1 with some forward velocity and comes to rest at an overhanging distance d_2 less than c as diagramed in Fig. 3, the initial velocity of the car can be determined by substituting d_1 in Eq 13.

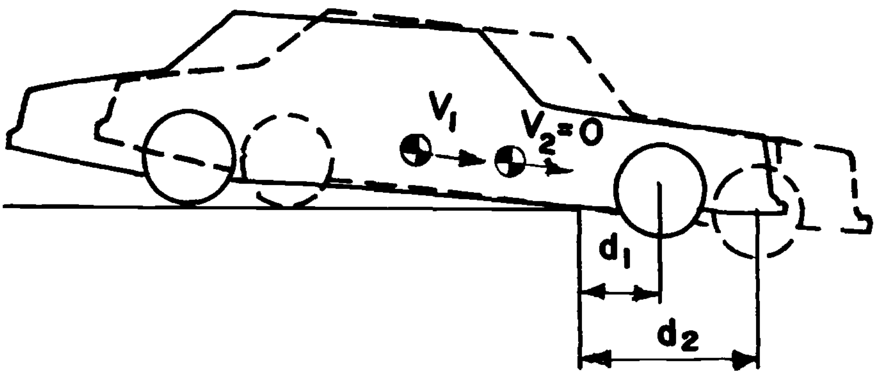

FIG. 3—*Forward velocity reduces to zero at distance* d *less than* c.

Velocity Reaches Zero at d *Greater Than* c

Referring to Fig. 4, assume the car falls to the edge of an embankment at a distance d_1 less than c and slides to a stop at a distance d_2 greater than c. At the distance d_1, the contact load is

$$P_1 = \frac{Wb}{B - d_1}$$

and at the distance c is equal to the weight W, so the average load in the distance d_1 to c is

$$\frac{1}{2}\left(\frac{Wb}{B - d_1} + W\right) = \frac{W}{2}\left(\frac{b}{B - d_1} + 1\right)$$

In the distance between c and d_2, the contact load is W. The retarding force acting in the distance d_1 to d_2 for a drag factor μ is

$$\frac{W\mu}{2}\left(\frac{b}{B - d_1} + 1\right) + W\mu$$

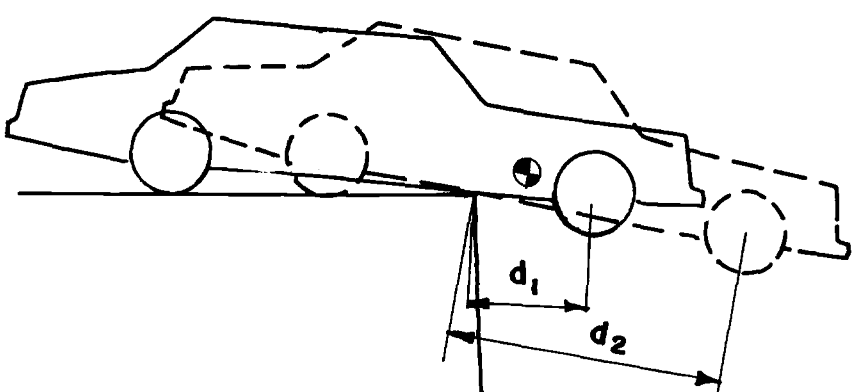

FIG. 4—*Forward velocity reduces to zero at distance* d *greater than* c.

and the deceleration of the car is

$$\alpha = \frac{\mu g}{2}\left(\frac{b}{B - d_1} + 1\right) + \mu g = \frac{\mu g}{2}\left(\frac{b}{B - d_1} + 3\right) \tag{14}$$

Then, the initial velocity of the car is

$$V = \sqrt{\mu g\left(\frac{b}{B - d_1} + 3\right)\left(d_2 - d_1\right)} \tag{15}$$

and the time of sliding between d_1 and d_2 is

$$t_{1\rightarrow 2} = 2\sqrt{\frac{d_2 - d_1}{\mu g\left(\frac{b}{B - d_1} + 3\right)}} \tag{16}$$

When the distance d is larger than c, the overhang of the mass center produces a moment to generate a pitching motion. At d_2 larger than c, the moment is approximately $T = W(d_2 - c)$. The mass moment of inertia of the car at $d = c$ is approximately I, and at d_2 is approximately $I_2 = I + M(d_2 - c).^2$ For average values of overturning moment and moment of inertia, the pitch acceleration is

$$\alpha = \frac{W/2(d_2 - c)}{1/2[I + I + M(d_2 - c)^2]} = \frac{W(d_2 - c)}{2I + M(d_2 - c)^2} \tag{17}$$

For the case of $V = 0$ at d_2, the linear velocity at $d = c$ is

$$V_c = \sqrt{2aS} = \sqrt{2\mu g(d_2 - c)} \tag{18}$$

The time of sliding motion between c and d_2 is

$$t_{c\rightarrow d_2} = \sqrt{\frac{2(d_2 - c)}{\mu g}} \tag{19}$$

The pitch velocity at c is negligibly small, so the pitch velocity at d_2 is

$$\omega_2 = \alpha t_{c\rightarrow d_2} = \frac{W(d_2 - c)}{2I + M(d_2 - c)^2}\sqrt{\frac{2(d_2 - c)}{\mu g}} \tag{20}$$

The pitch displacement at $d = c$ is $\theta_c = h/(B - c)$ and the pitch displacement at d_2 is

$$\begin{aligned}\theta_2 = \theta_c + \frac{\alpha t^2_{c\rightarrow d_2}}{2} &= \frac{h}{B - c} + \frac{1}{2}\left[\frac{W(d_2 - c)}{2I + M(d_2 - c)^2} \cdot \frac{2(d_2 - c)}{\mu g}\right] \\ &= \frac{h}{B - c} + \frac{W(d_2 - c)^2}{\mu g[2I + M(d_2 - c)^2]}\end{aligned} \tag{21}$$

If θ_2 is less than $\theta = \mu$ as diagramed in Fig. 5, the car remains at $d = d_2$ for a dwell period until $\theta = \mu$ when sliding motion will begin again. At the start of the dwell period, the linear velocity is zero but the car has a pitch velocity given by Eq 20. The pitching moment acting on the car is $T = W(d_2 - c)$ and the mass moment of inertia I_2 is given above, so the angular acceleration of pitch is

$$\alpha_{\theta_2 \to \mu} = \frac{T}{I_2} = \frac{W(d_2 - c)}{I + M(d_2 - c)^2} \tag{22}$$

At the start of the sliding motion after the dwell period, the pitch velocity is

$$\omega_\mu = \sqrt{\omega_2^2 + 2\alpha_{\theta_2 \to \mu}(\mu - \theta_2)} \tag{23}$$

The dwell time is

$$t = \sqrt{\frac{2(\mu - \theta_2)}{\alpha_{\theta_2 \to \mu}}} \tag{24}$$

Beginning at incipient sliding motion when $\theta = \mu$, it is convenient in problem solving to assume a small pitch displacement to an angle θ_3. The angular acceleration is given approximately by Eq 22 and the angular velocity at $\theta = \mu$ is given by Eq 23, so at θ_3 the angular velocity is

$$\omega_3 = \sqrt{\omega_\mu^2 + 2\alpha(\theta_3 - \theta_\mu)} \tag{25}$$

The time of motion between θ_μ and θ_3 is

$$t_{\theta_\mu \to \theta_3} = \frac{2(\theta_3 - \theta_\mu)}{\omega_\mu + \omega_3} \tag{26}$$

Now at θ_3 there is an accelerating force $W \sin \theta_3$ and a decelerating force $\mu W \cos \theta_3$ acting to give a net accelerating force $W(\sin \theta_3 - \mu \cos \theta_3)$, so the linear acceleration is

$$a_3 = g(\sin \theta_3 - \mu \cos \theta_3) \tag{27}$$

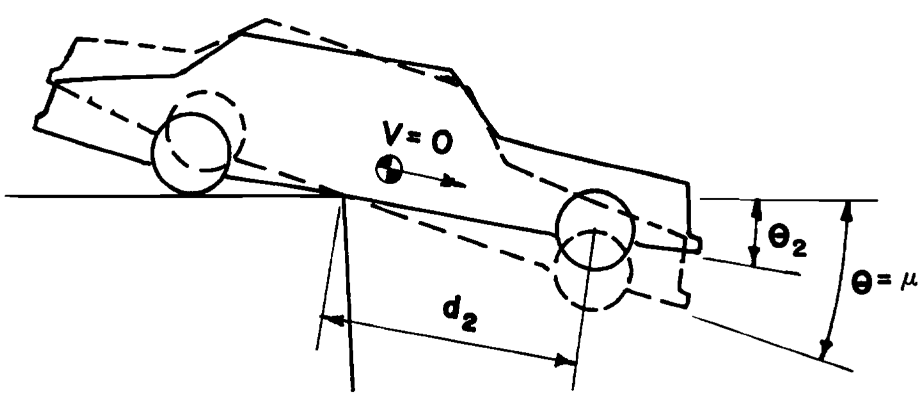

FIG. 5—*Condition in which* V $= 0$ *and* d_2 *is greater than* c *to produce a pitching moment.*

The linear velocity is

$$V_3 = a_3 t_{\theta_\mu \to \theta_3} \tag{28}$$

and the distance d_3 corresponding to the angular displacement θ_3 is

$$d_3 = d_2 + \frac{V_3}{2} t_{\theta_\mu \to \theta_3} \tag{29}$$

The direction of linear motion can be taken to be in the direction of the displacement angle θ_3.

A solution for subsequent motions of the car can be obtained by assuming incremental values of pitch displacement and solving for the remaining values until the limit of either case of the front of the car striking a landing base or of the rear wheels contacting the surface of the embankment. In either case the car has a linear velocity V_m and a pitch velocity ω_m.

Velocity Does Not Become Zero at d *Greater Than* c

The initial velocity and deceleration as a result of drag may be such that a car can fall to a contact at a distance d_1 either less than or greater than the distance c, but does not reach zero velocity before the front of the car lands on a surface below the embankment or the rear wheels strike the surface of the embankment. The initial velocity can be evaluated by substituting the measured value of d_1 for d in Eq 13. Subsequent motions may be of interest in some reconstruction problems.

If d_1 is less than c, the deceleration a between d_1 and some distance d_2 greater than c is given by Eq 14, and the velocity at d_2 is

$$V_2 = \sqrt{V_1^2 - 2a(d_2 - d_1)} \tag{30}$$

For d_2 greater than c, the angular acceleration as a result of overhang of the mass center is given by Eq 17 from which the angular velocity and angular pitch displacement are given by Eqs 20 and 21.

It has been shown in Ref *1* that if the initial velocity exceeds that given by Eq 9 there will be no contact between the underside of the car and the edge of the embankment as the car will be launched into a free-fall motion, and that at the velocity given by Eq 9 the car will only touch the edge of the embankment surface at a distance $d = 2B/3$. This means that the distance $d - c$ for d greater than c is relatively short.

Pitching Impulse Produced by Rear Wheels

When a car reaches a position as diagramed in Fig. 6, the rear wheels engage the ramp to impart a pitching motion. In a distance approximately equal to the radius of the wheel r, the rear axle is moved a distance equal to h to generate an impulsive force acting to increase the pitch velocity about an instantaneous center close to the front wheels.[2] The time to move from distance d_m to B is approximately $t_{m \to B} = r/V_m$, so the increase in pitch velocity is about

$$\Delta\omega = h/Bt = V_m h/Br \tag{31}$$

[2]Passenger cars have a dynamic index k^2/bc nearly equal to unity in the pitch mode in which k is the radius of gyration. If an impulsive force acts at the rear wheels, the instantaneous center is located at a distance i from the mass center and $i = k^2/b$. Then for a dynamic index of 1, $i = c$.

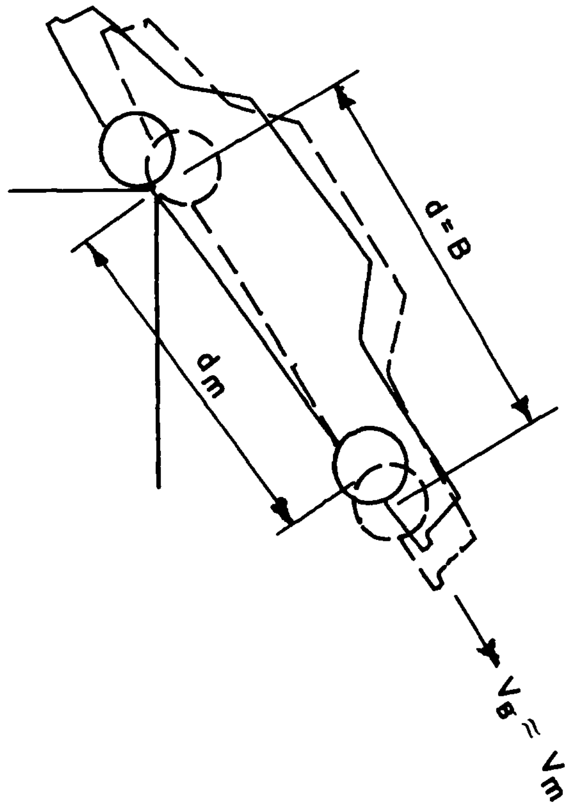

FIG. 6—*Rear wheels of a car in falling motion contact edge of embankment.*

Upon exiting the ramp, the car has a pitch velocity

$$\omega_B = \omega_m + \Delta\omega \tag{32}$$

and a pitch displacement

$$\theta_B = \theta_m + \frac{\omega_m + \omega_B}{2} t_{m \to B} \tag{33}$$

and a linear velocity of approximately V_m in the direction of θ_B.

Free-Fall Motions

When the car leaves the ramp at $d = B$ as diagramed in Fig. 7, the mass center follows a trajectory path in free-fall with the pitch displacement increasing with time.

Referring to Fig. 7, there is a component of velocity V_B' caused by pitch which is approximately

$$V_B' = \omega_B c \tag{34}$$

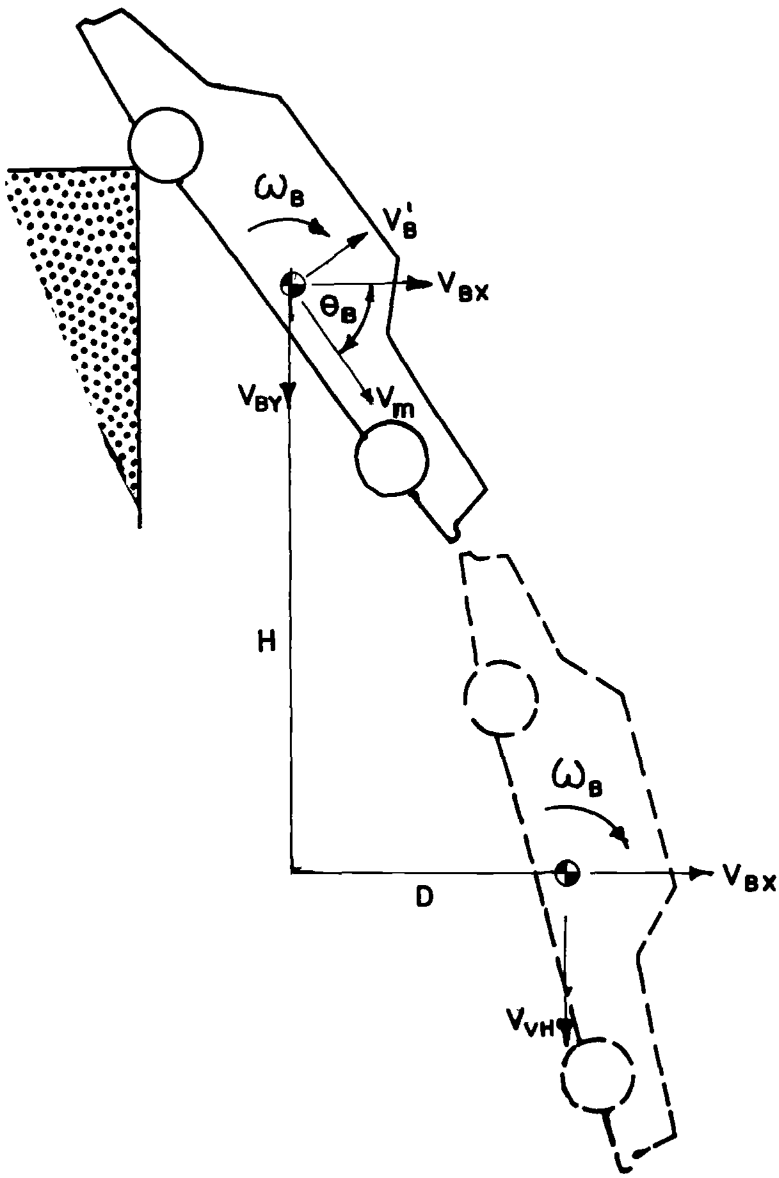

FIG. 7—*Car in free-fall motion after exit from embankment.*

in concert with the linear velocity V_B. The velocity component in the horizontal direction is

$$V_{BX} = V_m \cos \theta_B + V_B' \sin \theta_3 \qquad (35)$$

and in the vertical direction

$$V_{BY} = -V_m \sin \theta_3 + V_B' \cos \theta_3 \qquad (36)$$

In a fall distance H, the vertical component of velocity is

$$V_{VH} = \sqrt{V_{BY}^2 + 2gH} \qquad (37)$$

while the horizontal component of velocity remains equal to V_{BX}.

The horizontal displacement D is

$$D = V_{BX}t_F \qquad (38)$$

in which

$$t_F = \frac{2H}{V_{BY} + V_{VH}}$$

and in which time of free-fall the pitch displacement is

$$\theta_H = \theta_B + \dot{\theta}_B t_F \qquad (39)$$

Motions Upon Landing from Low-Speed Launch

The pitch displacement in low-speed launches can become quite large. The pitch velocity combined with the relatively large pitch displacement upon exiting a ramp at low speeds could cause the vehicle to land on its top side if the free-fall distance is large enough. It is possible that after the rear wheels leave the ramp or before the rear wheels contact the ramp as diagramed in Fig. 6, that the front of the car could strike a landing surface whereupon the inertia of pitch motion could cause the car to flip over on its top.

Referring to Fig. 8, assume that upon landing the vehicle has a pitch velocity ω_A and linear velocities V_A and V_A' in Position A. Also assume that the force of landing acting at the front of the vehicle is the only external impulse. Taking moments of momentum about Point 0

$$I\omega_A + MLV_A' + ML'V_A = I\omega_B + MLV_B$$

but

$$V_B = \omega_B L$$

so

$$I\omega_A + MLV_A' + ML'V_A = I\omega_B + ML^2\omega_B$$

$$= \omega_B(I + ML^2)$$

from which

$$\omega_B = \frac{I\omega_A + MLV_A' + ML'V_A}{I + ML^2} \qquad (40)$$

Assume for minimum conditions that at Position C $V_c = \omega_c = 0$, so the kinetic energy is

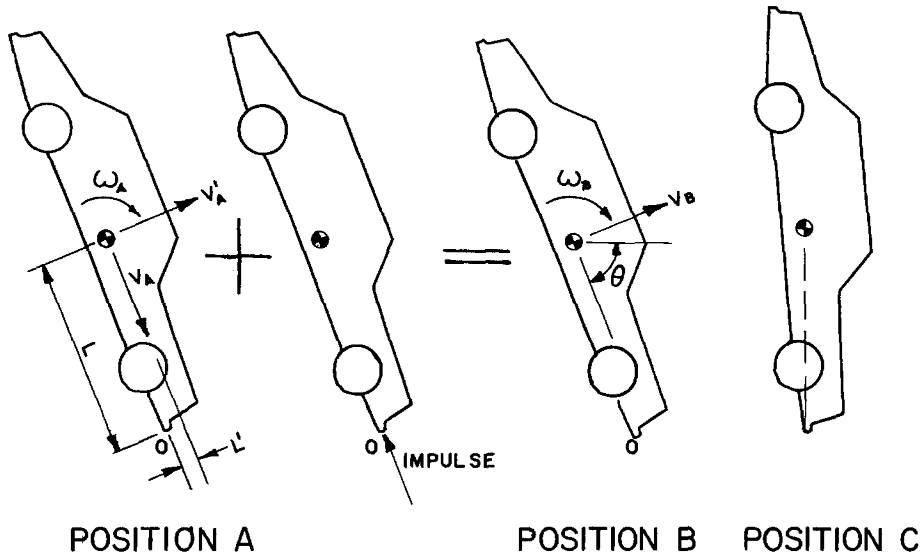

FIG. 8—*Motions of a car after landing from a free fall over an embankment.*

zero while the potential energy is approximately WL. At Position B, the kinetic energy is $1/2(I\omega_B^2 + MV_B^2)$ and the potential energy is $WL \sin \theta$. Equating,

$$\frac{I\omega_B^2}{2} + \frac{ML^2\omega_B^2}{2} + WL \sin \theta = WL$$

$$\omega_B^2(I + ML^2) = 2WL(1 - \sin \theta)$$

from which the minimum value of ω_B is

$$\omega_B = \sqrt{\frac{2WL(1 - \sin \theta)}{I + ML^2}} \quad (41)$$

Thus, if the value of ω_B is larger than that given by Eq 40, the car will flip over on its top.

Reference

[1] Le Fevre, W. F., "Speed Analysis of Passenger Cars in Free-Fall Launch Motions," *Journal of Forensic Sciences*, Vol. 31, No. 3, July 1986, pp. 886-902.

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